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On the Cesaro Summability with Respect to the Walsh–Kaczmarz System

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The Walsh system will be considered in the Kaczmarz rearrangement. We show that the maximal operator σ^* of the (C,1)-means of the Walsh–Kaczmarz–Fourier series is bounded from the dyadic Hardy space H^p into L^p for every $1/2 < p \leqslant 1$. From this it follows by standard arguments that σ^* is of weak type $(1, 1)$ and bounded from L^q into L^q if $1 < q \leqslant \infty$. © 2000 Academic Press

The a.e. convergence of the (C,1)-(Fejér) means of Walsh–Fourier series was investigated first by Fine [1]. He proved that these Cesaro means $\sigma_n f$ of an integrable function f converge a.e. to f as $n \to \infty$ if the Walsh system is taken in the Paley ordering. Schipp [5] considered the maximal operator $\sigma^* f := \sup_n |\sigma_n f|$ and showed that σ^* is of weak type (1,1). From this it follows by standard argument also the a.e. convergence. Since $\sigma^*: L^\infty \to L^\infty$ is bounded, Schipp's result implies by interpolation also the boundedness of $\sigma^*: L^p \to L^p$ $(1 < p \leqslant \infty)$. This fails to hold for $p = 1$ but Fujii [2] proved that σ^* is bounded from the dyadic Hardy space H^1 to L^1 (see also Simon [6]). Fujii's theorem was extended by Weisz [10] to H^p spaces, namely that $\sigma^*: H^p \to L^p$ $(1/2 < p \leqslant 1)$ is bounded.

If the Walsh system is taken in the Kaczmarz ordering, then the analogue of the statement of Schipp is due to Gát [3]. Moreover, he proved an (H^1, L^1)-like estimation, i.e., that $\|\sigma^* f\|_1 \leqslant C \, |||f|||_{H^1}$ $(f \in H^1)$.

In the present paper the above mentioned result of Weisz will be proved for the Walsh–Kaczmarz system. We show that σ^* is a so-called p-quasi local operator for every $1/2 < p \leqslant 1$. It is known (see Weisz [10]) that the p-quasi locality together with the L^∞-boundedness of σ^* implies that $\sigma^*: H^p \to L^p$ is bounded. The proof is based on the atomic structure of H^p. Furthermore, by known results on interpolation of operators we get the weak type (1,1) of σ^* and that $\sigma^*: L^q \to L^q$ $(1 < q \leqslant \infty)$ is also bounded.

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Finally, we extend the (H^p, L^p)-boundedness to every $0<p\leqslant 1$ if the maximal operator of the Cesaro means is considered only of order 2^n $(n=0, 1, ...)$.

2. DEFINITIONS AND NOTATIONS

In this section the most important definitions and notations are introduced. First of all we give a short summary of the basic concepts of the Walsh–Fourier analysis. Furthermore, we formulate some known statements, which will be cited in our investigations. For details see the book Schipp–Wade–Simon and Pál [4].

Let G be the so-called *dyadic group*, i.e., the set of all sequences $x=(x_k, k\in\mathbf{N})$ with terms $x_k\in\{0, 1\}$ $(k\in\mathbf{N}:=\{0, 1, ...\})$. The group operation $\dot{+}$ in G is the coordinatewise addition modulo 2, i.e., if $x=(x_k, k\in\mathbf{N})$, $y=(y_k, k\in\mathbf{N})\in G$ then $x\dot{+}y:=(x_k\oplus y_k, k\in\mathbf{N})$, where $a\oplus b$ denotes the addition modulo 2 of $a, b\in\mathbf{N}$.

The topology of G is determined by the *intervals* of G, that is, by the sets

$$I_n(x):=\{y=(y_k, k\in\mathbf{N})\in G: y_0=x_0, ..., y_{n-1}=x_{n-1}\}$$
$$(x\in G, 0<n\in\mathbf{N}).$$

Let $I_0:=G$ and $I_n:=I_n(0)$ $(n\in\mathbf{N})$, where $0:=(0, k\in\mathbf{N})\in G$ is the null element in G. Then G is a compact Abelian group. We consider the normalized Haar measure in G. The symbol L^p $(0<p\leqslant\infty)$ will denote the usual Lebesgue space of real-valued functions f defined on G with the norm (or quasinorm) $\|f\|_p:=(\int_G |f|^p)^{1/p}$ $(p<\infty)$, $\|f\|_\infty:=\text{ess sup}\,|f|$.

To the description of the characters w_n $(n\in\mathbf{N})$ of G let the functions r_k $(k\in\mathbf{N})$ be defined as $r_k(x):=(-1)^{x_k}$ $(x\in G)$. Then $(w_n, n\in\mathbf{N})$—the so-called Walsh–Paley system—is the product system generated by $(r_n, n\in\mathbf{N})$. Namely, if $n\in\mathbf{N}$ and $n=\sum_{k=0}^{\infty} n_k 2^k$ $(n_k=0, 1$ $(k\in\mathbf{N}))$ is the binary representation of n then

$$w_n=\prod_{k=0}^{\infty} r_k^{n_k}.$$

The functions

$$D_n:=\sum_{k=0}^{n-1} w_k, \qquad K_n:=\frac{1}{n}\sum_{k=1}^{n} D_k \qquad (n=1, 2, ...)$$

are the exact analogues of the well-known (trigonometric) *kernel functions* of Dirichlet's and Fejér's type, respectively. These functions have some good properties, useful also in the following investigations. First we mention a simple result with respect to Dirichlet kernels, which plays a central role in the Walsh–Fourier analysis:

$$D_{2^n}(x) = \begin{cases} 2^n & (x \in I_n) \\ 0 & (x \in G \setminus I_n) \end{cases} \qquad (n \in \mathbf{N}). \tag{1}$$

There is a strong connection between K_n $(n = 1, 2, ...)$ and D_{2^s} $(s \in \mathbf{N})$. Namely, the next relations hold for all $x \in G$ and $s \in \mathbf{N}$,

$$0 \leqslant K_{2^s}(x) = \tfrac{1}{2}\left(2^{-s} D_{2^s}(x) + \sum_{l=0}^{s} 2^{l-s} D_{2^s}(x \dotplus e_l)\right), \tag{2}$$

$$|K_l(x)| \leqslant \sum_{t=0}^{s} 2^{t-s-1} \sum_{i=t}^{s} (D_{2^i}(x) + D_{2^i}(x \dotplus e_t)) \qquad (2^s \leqslant l < 2^{s+1}), \tag{3}$$

where $e_l \in G$ $(l \in \mathbf{N})$ is determined by $(e_l)_k = 0$ $(k \neq l)$ and $(e_l)_l = 1$ $(k \in \mathbf{N})$. We remark that K_n's are uniformly L^1-bounded, i.e.,

$$\sup_n \|K_n\|_1 < \infty. \tag{4}$$

In this note the so-called Kaczmarz rearrangement $(\Psi_n, n \in \mathbf{N})$ (called *Walsh–Kaczmarz system*) of $(w_n, n \in \mathbf{N})$ will be investigated. The functions Ψ_n $(n \in \mathbf{N})$ are defined in the following way. If $0 < n \in \mathbf{N}$ then there is a unique $s \in \mathbf{N}$ such that the binary representation of n is of the form $n = 2^s + \sum_{k=0}^{s-1} n_k 2^k$. Then let

$$\Psi_n(x) := r_s(x) \prod_{k=0}^{s-1} r_{s-k-1}^{n_{k(x)}} \qquad (x \in G).$$

Furthermore, let $\Psi_0 := w_0$. It is not hard to see that $\Psi_{2^m} = w_{2^m} = r_m$ and $\{\Psi_k : k = 2^m, ..., 2^{m+1} - 1\} = \{w_k : k = 2^m, ..., 2^{m+1} - 1\}$ $(m \in \mathbf{N})$. Finally, if

$$\tau_s(x) := (x_{s-1}, x_{s-2}, ..., x_1, x_0, x_s, x_{s+1}, ...) \in G \qquad (x \in G)$$

then

$$\Psi_n(x) = r_s(x)\, w_{n-2^s}(\tau_s(x)) \qquad (x \in G).$$

We remark that by (1) we get $D_{2^j}(\tau_j(x)) = D_{2^j}(x)$ $(j \in \mathbf{N}, x \in G)$.

It is clear that $(\Psi_n, n \in \mathbf{N})$ is also a complete orthonormal system. If $f \in L^1$ then let $\hat{f}(k) := \int_G f \Psi_k$ $(k \in \mathbf{N})$ be the kth Fourier coefficient of f

with respect to $(\Psi_n, n \in \mathbf{N})$. Denote by $\sigma_n f$ $(0 < n \in \mathbf{N})$ the nth *Fejér mean* of f, i.e., let

$$\sigma_n f := \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) \hat{f}(k)\, \Psi_k.$$

Then $\sigma_n f(x) = \int_G f(t)\, \mathscr{K}_n(x \dotplus t)\, dt$ $(x \in G)$, where the nth *Fejér kernel* with respect to the Walsh–Kaczmarz system is given by

$$\mathscr{K}_n := \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) \Psi_k.$$

A simple calculation shows (see also Skvortsov [8]) that for $x \in G$ and $2^s + m$ $(s \in \mathbf{N},\ m = 0, ..., 2^s - 1)$

$$(2^s + m)\, \mathscr{K}_{2^s+m}(x) = 1 + \sum_{j=0}^{s-1} 2^j D_{2^j}(x) + m\, D_{2^s}(x) + \sum_{j=0}^{s-1} 2^j r_j(x)\, K_{2^j}(\tau_j(x)) + m r_s(x)\, K_m(\tau_s(x)). \tag{6}$$

By (1), (4) and (6) it is clear that also $\mathscr{K}_n$'s are L^1-bounded:

$$\sup_n \|\mathscr{K}_n\|_1 < \infty. \tag{7}$$

In this note the *maximal operator* σ^* will be investigated, where

$$\sigma^* f := \sup_n |\sigma_n f| \qquad (f \in L^1).$$

The estimation (7) implies obviously that $\sigma^*\colon L^\infty \to L^\infty$ is bounded.

3. HARDY SPACES

Hardy spaces can be defined in various manner. (For details see e.g., the book of Weisz [11].) To this end let the *maximal function* of $f \in L^1$ be given by

$$f^*(x) = \sup_n 2^n \left| \int_{I_n(x)} f \right| \qquad (x \in G)$$

and introduce the martingale *Hardy spaces* for $0 < p < \infty$ as follows: denote H^p the space of f's for which $\|f\|_{H^p} := \|f^*\|_p < \infty$. It is well-known that for $1 < p < \infty$ the space H^p is nothing else than L^p.

For $0<p\leqslant 1$ the atomic decomposition is a useful characterization of H^p. To demonstrate this we give first the concept of atoms as follows: a function $a\in L^\infty[0,1)$ is called a *p-atom* if either a is identically equal to 1 or if there exists a dyadic interval $I=I_N(x)$ for some $N\in\mathbf{N}$, $x\in G$ such that

$$\operatorname{supp} a\subset I, \qquad \|a\|_\infty\leqslant 2^{N/p} \qquad \text{and} \qquad \int_G a=0. \tag{8}$$

We shall say that a is supported on I. Then if a function f belongs to H^p then there exist a sequence $(a_k, k=0, 1, ...)$ of p-atoms and a sequence $(\mu_k, k=0, 1, ...)$ of real numbers such that $\sum_{k=0}^\infty |\mu_k|^p<\infty$ and

$$f=\sum_{k=0}^\infty \mu_k a_k. \tag{9}$$

Moreover, the following equivalence of norms holds,

$$c_p\,\|f\|_{H^p}\leqslant \inf\left(\sum_{k=0}^\infty |\mu_k|^p\right)^{1/p}\leqslant C_p\,\|f\|_{H^p} \qquad (f\in H^p), \tag{10}$$

where the infimum is taken over all decompositions of f of the form (9). (Here and later c_p, C_p, C will denote positive constants depending at most on p, although not always the same in different occurrences.)

A sublinear operator T which maps H^p into the collection of measurable functions defined on G will be called *p-quasi local* if there exists a constant C_p such that

$$\int_{G\setminus I} |Ta|^p\leqslant C_p \tag{11}$$

for every p-atom a supported on I. Assume the L^∞-boundedness of T, i.e., that $\|Tf\|_\infty\leqslant C\,\|f\|_\infty$ $(f\in L^\infty)$. Then it is known (see Weisz [10] or Simon [7]) that for T to be bounded from H^p to L^p it is sufficient that T is p-quasi local.

4. CESARO SUMMABILITY

The Walsh–Kaczmarz system was studied by many authors. So e.g., Schipp [5] proved that this system is a convergence system (see also

Wo-Sang Young [12]). In Skvortsov [8, 9] the L^1-convergence and the uniform convergence of Fejér means was investigated. Gát [3] showed that $\lim_n \sigma_n f(x) = f(x)$ (a.e. $x \in G$, $f \in L^1$). Moreover, he proved that $\sigma^*: L^q \to L^q$ is bounded for all $1 < q \leqslant \infty$ and of weak type (1,1). Furthermore, it is also proved the estimation $\|\sigma^* f\|_1 \leqslant C \, \||f|\|_{H^1}$ ($f \in H^1$). In this connection we refer to Fujii [2] (see also Simon [6]), namely, that the analogous estimation with f instead of $|f|$ holds with respect to the Walsh–Paley system. This last result was extended by Weisz [10] to H^p ($1/2 < p$).

In the present work we give the analogue of Weisz's result for the Walsh–Kaczmarz system. Namely, the following theorem will be proved.

THEOREM 1. *Let* $1/2 < p \leqslant 1$. *Then* $\sigma^*: H^p \to L^p$ *is bounded.*

Applying known results on interpolation of operators (see Weisz [10] or Simon [7]) we get the next

COROLLARY 1. *If* $1 < q \leqslant \infty$ *then* $\sigma^*: L^q \to L^q$ *is bounded. Moreover,* σ^* *is of weak type* (1,1).

From Corollary 1 it follows by standard arguments

COROLLARY 2. *If* $f \in L^1$ *then* $\lim_n \sigma_n f(x) = f(x)$ *for a.e.* $x \in G$.

If we consider the maximal operator of the Fejér means of order 2^n ($n \in \mathbf{N}$) then Theorem 1 can be extended to every $0 < p \leqslant 1$. In other words the next theorem is true.

THEOREM 2. *Let* $0 < p \leqslant 1$. *Then there exists a constant* C_p *such that* $\|\sup_n |\sigma_{2^n} f|\|_p \leqslant C_p \|f\|_{H^p}$ *for every* $f \in H^p$.

5. PROOF OF THEOREMS

Taking into account the previous observations it is enough to prove that the maximal operators in question are p-quasi local for all $1/2 < p \leqslant 1$ or $0 < p \leqslant 1$, i.e., (11) holds for $T := \sigma^*$ or $Tf := \sup_n |\sigma_{2^n} f|$ ($f \in L^1$). To this end let a be a p-atom. It can be supposed that a is supported on I_N for some $N \in \mathbf{N}$. That is, $\|a\|_\infty \leqslant 2^{N/p}$ and $\int_G a = \int_{I_N} a = 0$. This implies that $\hat{a}(k) = 0$ ($k = 0, ..., 2^s - 1$) therefore $\sigma_{2^s + m} a = 0$ if $s = 0, ..., N - 1$, $m = 0, ..., 2^s - 1$. Hence assume $N \leqslant s \in \mathbf{N}$, $m = 0, ..., 2^s - 1$ and $x \in G \backslash I_N$. Then we get by (6) and (1)

$$\sigma_{2^s+m}a(x)$$

$$= \frac{1}{2^s+m}\int_{I_N} a(t)\left(1+\sum_{j=0}^{s-1} 2^j D_{2^j}(x \dot{+} t) + m\, D_{2^s}(x \dot{+} t)\right.$$

$$\left. + \sum_{j=0}^{s-1} 2^j r_j(x \dot{+} t)\, K_{2^j}(\tau_j(x \dot{+} t)) + m r_s(x \dot{+} t)\, K_m(\tau_s(x \dot{+} t))\right) dt$$

$$= \frac{1}{2^s+m}\int_{I_N} a(t)\left(\sum_{j=N+1}^{s-1} 2^j r_j(x \dot{+} t)\, K_{2^j}(\tau_j(x \dot{+} t))\right.$$

$$\left. + m r_s(x \dot{+} t)\, K_m(\tau_s(x \dot{+} t))\right) dt$$

$$= \frac{1}{2^s+m}\sum_{j=N+1}^{s-1} 2^j \int_{I_N} a(t)\, r_j(x \dot{+} t)\, K_{2^j}(\tau_j(x \dot{+} t))\, dt$$

$$+ \frac{m}{2^s+m}\int_{I_N} a(t)\, r_s(x \dot{+} t)\, K_m(\tau_s(x \dot{+} t))\, dt =: \sigma_n^{(1)}(x) + \sigma_n^{(2)}(x),$$

where $n := 2^s + m$.

Let us examine first $\sigma_n^{(1)}(x)$. Applying (2) it follows that

$$\sigma_n^{(1)}(x) = \frac{1}{2^s+m}\sum_{j=N+1}^{s-1} 2^{j-1}\int_{I_N} a(t)\, r_j(x \dot{+} t)\left(2^{-j} D_{2^j}(\tau_j(x \dot{+} t))\right.$$

$$\left. + \sum_{l=0}^{j} 2^{l-j} D_{2^j}(\tau_j(x \dot{+} t) \dot{+} e_l)\right) dt.$$

Since $D_{2^j}(\tau_j(x \dot{+} t)) = D_{2^j}(x \dot{+} t) = 0$ if $t \in I_N$ thus

$$\sigma_n^{(1)}(x) = \frac{1}{2^s+m}\sum_{j=N+1}^{s-1} 2^{j-1}\sum_{l=0}^{j} 2^{l-j}\int_{I_N} a(t)\, r_j(x \dot{+} t)\, D_{2^j}(\tau_j(x \dot{+} t) \dot{+} e_l))\, dt$$

$$= \frac{1}{2^s+m}\sum_{j=N+1}^{s-1}\sum_{l=0}^{j} 2^{l-1}\int_{I_N} a(t)\, r_j(x \dot{+} t)\, D_{2^j}(\tau_j(x \dot{+} t) \dot{+} e_l))\, dt.$$

Let $v = 0, ..., N-1$ such that $x \in I_v \backslash I_{v+1}$. If $j = N+1, ..., s-1$ and z_k ($k \in \mathbf{N}$) denotes the kth coordinate of $\tau_j(x \dot{+} t)$ then $z_0 = x_{j-1} \oplus t_{j-1}, ...,$ $z_{j-N-1} = x_N \oplus t_N$, $z_{j-N} = x_{N-1}, ..., z_{j-v-2} = x_{v+1}$, $z_{j-v-1} = 1$, $z_{j-v} = 0, ...,$ $z_{j-1} = 0$, $z_j = x_j \oplus t_j, ...$. From this it follows by (1) that $D_{2^j}(\tau_j(x \dot{+} t) \dot{+} e_l) \neq 0$ ($t \in I_N$) iff $l = j - v - 1$ and $x_{v+1} = \cdots = x_{N-1} = 0$. Let $I_{N,v}$ be the set of such x's, then the measure of $I_{N,v}$ is 2^{-N}. Thus we get the next estimation

$$|\sigma_n^{(1)}(x)| \leqslant \frac{1}{2^s+m} \left| \sum_{j=N+1}^{s-1} 2^{j-v-2} \int_{I_N} a(t)\, r_j(x \dotplus j)\, D_{2^j}(\tau_j(x \dotplus t) \dotplus e_{j-v-1})\, dt \right|$$

$$\leqslant \frac{1}{2^s+m} \frac{1}{2^{v+2}} \sum_{j=N+1}^{s-1} 2^j \|a\|_\infty \leqslant C \frac{\|a\|_\infty}{2^v}.$$

This implies

$$\int_{G \setminus I_N} \sup_n |\sigma_n^{(1)}(x)|^p\, dx = \sum_{v=0}^{N-1} \int_{I_{N,v}} \sup_n |\sigma_n^{(1)}(x)|^p\, dx$$

$$\leqslant C_p \|a\|_\infty^p \sum_{v=0}^{N-1} \frac{1}{2^{pv}} \cdot \frac{1}{2^N} \leqslant C_p$$

for all $0<p\leqslant 1$. We recall that $\sigma_{2^s} a(x) = \sigma_{2^s}^{(1)}(x)$ $(x \in G \setminus I_N)$, therefore the last estimation proves Theorem 2.

Now, let $s \in \mathbf{N}$, $s \geqslant N$; $l=0, ..., s$ and $m=2^{l-1}, ..., 2^l-1$. Then by (3)

$$|K_m(\tau_s(x \dotplus t))| \leqslant \sum_{j=0}^{l-1} 2^{j-l} \sum_{i=j}^{l-1} (D_{2^i}(\tau_s(x \dotplus t)) + D_{2^i}(\tau_s(x \dotplus t) \dotplus e_j)),$$

therefore

$$|\sigma_{2^s+m}^{(2)}(x)| \leqslant \frac{m\, \|a\|_\infty}{2^s+m} \sum_{j=0}^{l-1} 2^{j-l} \sum_{i=j}^{l-1} \int_{I_N} (D_{2^i}(\tau_s(x \dotplus t)) + D_{2^i}(\tau_s(x \dotplus t) \dotplus e_j))\, dt$$

$$= \frac{m\, \|a\|_\infty}{2^s+m} \sum_{i=0}^{l-1} \sum_{j=0}^{i} 2^{j-l} \int_{I_N} (D_{2^i}(\tau_s(x \dotplus t)) + D_{2^i}(\tau_s(x \dotplus t) \dotplus e_j))\, dt$$

$$\leqslant C \frac{\|a\|_\infty}{2^s} \left(\sum_{i=0}^{l-1} 2^i \int_{I_N} D_{2^i}(\tau_s(x \dotplus t))\, dt \right.$$

$$\left. + \sum_{i=1}^{l-1} \sum_{j=0}^{i-1} 2^j \int_{I_N} D_{2^i}(\tau_s(x \dotplus t) \dotplus e_j)\, dt \right)$$

$$=: \sigma_{2^s+m}^{(20)}(x) + \sigma_{2^s+m}^{(21)}(x).$$

Let $x \in I_v \setminus I_{v+1}$ $(v=0, ..., N-1)$ and investigate first $\sup_{s \geqslant N;\, m=0, ..., 2^s-1} \times \sigma_{2^s+m}^{(20)}(x)$ as follows:

$$\sup_{s \geqslant N;\, m=0, ..., 2^s-1} \sigma^{(20)}_{2^s+m}(x) = \sup_{s \geqslant N} \max_{l=0, ..., s} \max_{m=2^{l-1}, ..., 2^l-1} \sigma^{(20)}_{2^s+m}(x)$$

$$\leqslant C \|a\|_\infty \sup_{s \geqslant N} \frac{1}{2^s} \left(\sum_{i=0}^{s-N} 2^i \int_{I_N} D_{2^i}(\tau_s(x \dot{+} t))\, dt + \sum_{i=s-N+1}^{s-v-1} 2^i \int_{I_N} D_{2^i}(\tau_s(x \dot{+} t))\, dt + \sum_{i=s-v}^{s-1} 2^i \int_{I_N} D_{2^i}(\tau_s(x \dot{+} t))\, dt \right).$$

If $t \in I_N$ and $i = 0, ..., s-N$ then (see the coordinates of $\tau_s(x \dot{+} t)$ above)

$$D_{2^i}(\tau_s(x \dot{+} t)) = \begin{cases} 2^i & \text{if} \quad t = (0, ..., 0, t_N, ..., t_{s-i-1}, x_{s-i}, ..., x_{s-1}, t_s, ...) \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $I^{(s)}_{N,i}(x)$ the set of t's of the form

$$t = (0, ..., 0, t_N, ..., t_{s-i-1}, x_{s-i}, ..., x_{s-1}, t_s, ...).$$

Then the measure of $I^{(s)}_{N,i}(x)$ is 2^{-N-i} and

$$\sup_{s \geqslant N} \frac{1}{2^s} \sum_{i=0}^{s-N} 2^i \int_{I_N} D_{2^i}(\tau_s(x \dot{+} t))\, dt$$

$$\leqslant C \sup_{s \geqslant N} \frac{1}{2^s} \sum_{i=0}^{s-N} 2^i \int_{I^{(s)}_{N,i}(x)} D_{2^i}(\tau_s(x \dot{+} t))\, dt$$

$$\leqslant C \sup_{s \geqslant N} \frac{1}{2^s} \sum_{i=0}^{s-N} 2^i \frac{2^i}{2^{N+i}} \leqslant \frac{1}{2^{2N}}.$$

If $x \in I_v \backslash I_{v+1}$, $t \in I_N$ and $i = s-N+1, ..., s-v-1$ then $D_{2^i}(\tau_s(x \dot{+} t)) = 2^i$ iff $x_{N-1} = \cdots = x_{s-i} = 0$ and $t \in I^{(s)}_{N,s-1-N}(x)$. Otherwise $D_{2^i}(\tau_s(x \dot{+} t)) = 0$. Therefore if $l = v, ..., N-1$ and $J^{(l)}_{N,v}$ stands for the set of z's in G such that $z = (0, ..., 0, 1, z_{v+1}, ..., z_{l-1}, 0, ..., 0, z_N, z_{N+1}, ...)$ then $I_v \backslash I_{v+1} = \bigcup_{l=v}^{N-1} J^{(l)}_{N,v}$ and for $x \in J^{(l)}_{N,v}$ ($l = v, ..., N-1$) we have

$$\sup_{s \geqslant N} \frac{1}{2^s} \sum_{i=s-N+1}^{s-v-1} 2^i \int_{I_N} D_{2^i}(\tau_s(x \dot{+} t))\, dt \leqslant C \sup_{s \geqslant N} \frac{1}{2^s} \sum_{i=s-N+1}^{s-l-1} 2^i \frac{2^i}{2^s}$$

$$\leqslant C \sup_{s \geqslant N} \frac{1}{2^{2s}} 2^{2(s-l)} \leqslant C \frac{1}{2^{2l}}.$$

(We remark that the measure of $J^{(l)}_{N,v}$ is $2^{-v-N+l-2}$.)

If $x \in I_v \backslash I_{v+1}$, $i = s-v, ..., s-1$ and $t \in I_N$, then $\tau_s(x \dot{+} t) \notin I_i$ so by (1) $D_{2^i}(\tau_s(x \dot{+} t)) = 0$.

Summarizing the above facts it follows that

$$\begin{aligned}\int_{G\setminus I_N}(\sup_n \sigma_n^{(20)}(x))^p\,dx &\leqslant C_p\,\|a\|_\infty^p\,\frac{1}{2^{2pN}}+\sum_{v=0}^{N-1}\sum_{l=v}^{N-1}\int_{I^{(l)}_{N,v}}(\sup_n \sigma_n^{(20)}(x))^p\,dx\\ &\leqslant C_p\,\|a\|_\infty^p\,\frac{1}{2^{2pN}}+C_p\,\|a\|_\infty^p\sum_{v=0}^{N-1}\sum_{l=v}^{N-1}\frac{1}{2^{2pl}}\,\frac{1}{2^{v+N-l}}\\ &\leqslant C_p\,\|a\|_\infty^p\,\frac{1}{2^{2pN}}+C_p\,\|a\|_\infty^p\,\frac{1}{2^N}\sum_{v=0}^{N-1}\frac{1}{2^v}\sum_{l=v}^{N-1}\frac{1}{2^{l(2p-1)}}\\ &\leqslant C_p,\end{aligned}$$

if $1/2\leqslant p\leqslant 1$.

Now, we investigate $\sigma_{2^s+m}^{(21)}(x)$ for $x\in I_v\setminus I_{v+1}$ $(v=0,\ldots,N-1)$, $l=0,\ldots,$ $s\in\mathbf{N}$, $m=2^{l-1},\ldots,2^l-1$. Then

$$\begin{aligned}&\sup_{s\geqslant N}\ \max_{l=0,\ldots,s}\ \max_{m=2^{l-1},\ldots,2^l-1}\sigma_{2^s+m}^{(21)}(x)\\ &\quad\leqslant C\,\|a\|_\infty\sup_{s\geqslant N}\frac{1}{2^s}\left(\sum_{i=1}^{s-N}\sum_{j=0}^{i-1}2^j\int_{I_N}D_{2^i}(\tau_s(x\dot+t)\dot+e_j)\right)dt\\ &\quad+\sum_{i=s-N+1}^{s-v-1}\sum_{j=0}^{i-1}2^j\int_{I_N}D_{2^i}(\tau_s(x\dot+t)\dot+e_j)\,dt\\ &\quad+\sum_{i=s-v}^{s-1}\sum_{j=0}^{i-1}2^j\int_{I_N}D_{2^i}(\tau_s(x\dot+t)\dot+e_j)\,dt.\end{aligned}$$

As above (see the case $i=1,\ldots,s-N$) we have

$$\begin{aligned}&\sup_{s\geqslant N}\frac{1}{2^s}\sum_{i=1}^{s-N}\sum_{j=0}^{i-1}2^j\int_{I_N}D_{2^i}(\tau_x(x\dot+t)\dot+e_j)\,dt\\ &\quad=\sup_{s\geqslant N}\frac{1}{2^s}\sum_{i=1}^{s-N}\sum_{j=0}^{i-1}2^j\int_{I^{(s)}_{N,i}(x\dot+e_{s-j-1})}D_{2^i}((\tau_s(x\dot+t)\dot+e_j)\,dt\\ &\quad\leqslant C\sup_{s\geqslant N}\frac{1}{2^s}\sum_{i=1}^{s-N}\sum_{j=0}^{i-1}2^j\,\frac{2^i}{2^{N+i}}\leqslant C\,\frac{1}{2^{2N}}.\end{aligned}$$

Let $N\leqslant s\in\mathbf{N}$, $v=0,\ldots,N-1$ and decompose the double sum $\sum_{i=s-N+1}^{s-v-1}\sum_{j=0}^{i-1}2^j\int_{I_N}D_{2^i}(\tau_s(x\dot+t)\dot+e_j)\,dt$ in the following way:

$$\sum_{i=s-N+1}^{s-v-1} \sum_{j=0}^{i-1} 2^j \int_{I_N} D_{2^i}(\tau_s(x \dotplus t) \dotplus e_j)\, dt$$
$$= \sum_{i=s-N+1}^{s-v-1} \sum_{j=0}^{s-N-1} 2^j \int_{I_N} D_{2^i}(\tau_s(x \dotplus t) \dotplus e_j)\, dt$$
$$+ \sum_{i=s-N+1}^{s-v-1} \sum_{j=s-N}^{i-1} 2^j \int_{I_N} D_{2^i}(\tau_s(x \dotplus t) \dotplus e_j)\, dt.$$

Furthermore we suppose that $x \in J_{N,v}^{(l)}$ $(l = v, ..., N-1)$ which implies

$$\sup_{s \geqslant N} \frac{1}{2^s} \sum_{i=s-N+1}^{s-v-1} \sum_{j=0}^{s-N-1} 2^j \int_{I_N} D_{2^i}(\tau_s(x \dotplus t) \dotplus e_j)\, dt$$
$$= \sup_{s \geqslant N} \frac{1}{2^s} \sum_{i=s-N+1}^{s-v-1} \sum_{j=0}^{s-N-1} 2^j \int_{I_{N,s-N}^{(s)}(x \dotplus e_{s-j-1})} D_{2^i}(\tau_s(x \dotplus t) \dotplus e_j)\, dt$$
$$= \sup_{s \geqslant N} \frac{1}{2^s} \sum_{i=s-N+1}^{s-l-1} \sum_{j=0}^{s-N-1} 2^j \frac{2^i}{2^s} \leqslant C \sup_{s \geqslant N} \frac{1}{2^{2s}} \sum_{i=s-N+1}^{s-l-1} 2^i \sum_{j=0}^{s-N-1} 2^j$$
$$\leqslant C \sup_{s \geqslant N} \frac{1}{2^{2s}} 2^{s-l} 2^{s-N} \leqslant C \frac{1}{2^{l+N}}.$$

Now, for $N \leqslant s \in \mathbf{N}$, $v = 0, ..., N-1$ and $x \in I_v \backslash I_{v+1}$ the sum

$$\sum_{i=s-N+1}^{s-v-1} \sum_{j=s-N}^{i-1} 2^j \int_{I_N} D_{2^i}(\tau_s(x \dotplus t) \dotplus e_j)\, dt$$

will be investigated. To this end let $l = v, ..., N-1$ and $x \in J_{N,v}^{(l)}$, $t \in I_{N,s-N}^{(s)}$. If (as above) z_k $(k \in \mathbf{N})$ denotes the kth coordinate of $\tau_s(x \dotplus t)$ then $z_0 = \cdots = z_{s-l-2} = 0$, $z_{s-l-1} = 1$, $z_{s-l} = x_{s-l}, ..., z_{s-v+2} = x_{v+1}$, $z_{s-v-1} = 1$, $z_{s-v} = \cdots = z_{s-1} = 0$, $z_s = x_s \oplus t_s, ...$ This means that for $i = s-N+1, ..., s-l-1$ the $j(\leqslant i-1)$th coordinate of $\tau_s(x \dotplus t) \dotplus e_j$ is equal to 1 and so by (1) $D_{2^i}(\tau_s(x \dotplus t) \dotplus e_j) = 0$. On the other hand if $i \geqslant s-l$ then for $j = s-l-1$ $D_{2^i}(\tau_s(x \dotplus t) \dotplus e_j) = 2^i$, otherwise $D_{2^i}(\tau_s(x \dotplus t) \dotplus e_j) = 0$. Therefore

$$\sum_{i=s-N+1}^{s-v-1} \sum_{j=s-N}^{i-1} 2^j \int_{I_N} D_{2^i}(\tau_s(x \dotplus t) \dotplus e_j)\, dt$$
$$= \sum_{i=s-l}^{s-v-1} 2^{s-l-1} \int_{I_{N,s-N}^{(s)}} D_{2^i}(\tau_s(x \dotplus t) \dotplus e_{s-l-1})\, dt.$$

Furthermore, let $J_{N,v}^{(l,k)}$ $(k = v+1, ..., l-1)$ be the set of $u \in J_{N,v}^{(l)}$ such that $u_{l-1} = \cdots = u_k = 0$ and $u_{k-1} = 1$. The measure of $J_{N,v}^{(l,k)}$ is $2^{-v-N+k-2}$. If

$x \in J_{N,v}^{(l,k)}$ then $D_{2^i}(\tau_s(x \dotplus t) \dotplus e_{s-l-1}) = 2^i (i = s-l, ..., s-k)$ and if $i > s-k$ then $D_{2^i}(\tau_s(x \dotplus t) \dotplus e_{s-l-1}) = 0$. Thus

$$\sup_{s \geqslant N} \frac{1}{2^s} \sum_{i=s-l}^{s-v-1} 2^{s-l-1} \int_{I_{N,s-N}^{(s)}} D_{2^i}(\tau_s(x \dotplus t) \dotplus e_{s-l-1})\, dt$$
$$= \sup_{s \geqslant N} \frac{1}{2^s} \sum_{i=s-l}^{s-k} 2^{s-l-1} \int_{I_{N,s-N}^{(s)}} D_{2^i}(\tau_s(x \dotplus t) \dotplus e_{s-l-1})\, dt$$
$$= \sup_{s \geqslant N} \frac{1}{2^s} \sum_{i=s-l}^{s-k} 2^{s-l-1} \frac{2^i}{2^s} \leqslant C \frac{1}{2^{l+k}}.$$

We get by similar observations that for $x \in I_v \backslash I_{v+1}$ $(v = 0, ..., N-1)$, $i = s-v, ..., s-1$ and for $j = 0, ..., i-1$

$$\int_{I_N} D_{2^i}(\tau_s(x \dotplus t) \dotplus e_j)\, dt = \int_{I_{N,s-N}^{(s)}} D_{2^i}(\tau_s(x \dotplus t) \dotplus e_j)\, dt \neq 0$$

iff $x \in J_{N,v}^{(v)}$. From this it follows that

$$\sup_{s \geqslant N} \frac{1}{2^s} \sum_{i=s-v}^{s-1} \sum_{j=0}^{i-1} 2^j \int_{I_N} D_{2^i}(\tau_s(x \dotplus t) \dotplus e_j)\, dt$$
$$= \sup_{s \geqslant N} \frac{1}{2^s} \sum_{i=s-v}^{s-1} 2^{s-v-1} \frac{2^i}{2^s} \leqslant C \frac{1}{2^v}.$$

Thus the proof of Theorem 1 can be finished as follows:

$$\int_{G \backslash I_N} (\sup_n \sigma_n^{(21)}(x))^p\, dx$$
$$\leqslant C_p \left(\frac{\|a\|_\infty^p}{2^{2pN}} + \sum_{v=0}^{N-1} \sum_{l=v}^{N-1} \frac{\|a\|_\infty^p}{2^{p(l+N)}} \cdot \frac{1}{2^{v+n-l}} \right.$$
$$\left. + \sum_{v=0}^{N-1} \sum_{l=v}^{N-1} \sum_{k=v+1}^{l-1} \frac{\|a\|_\infty^p}{2^{p(l+k)}} \cdot \frac{1}{2^{v+N-k}} + \sum_{v=0}^{N-1} \frac{\|a\|_\infty}{2^{pv}} \cdot \frac{1}{2^N} \right)$$
$$\leqslant C_p \left(2^{(1-2p)N} + \frac{1}{2^{pN}} \sum_{v=0}^{N-1} \frac{1}{2^v} \sum_{l=v}^{N-1} 2^{(1-p)l} \right.$$
$$\left. + \sum_{v=0}^{N-1} \frac{1}{2^v} \sum_{l=v}^{N-1} \frac{1}{2^{pl}} \sum_{k=v+1}^{l-1} 2^{(1-p)k} + \sum_{v=0}^{N-1} \frac{1}{2^{pv}} \right) \leqslant C_p,$$

if $1/2 < p \leqslant 1$.

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